Week 2

14. Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$. (The sum of two subsets is defined in the exercises of Section 1.3.)

Proof. "$\subseteq$": $\forall \ v \in \text{span}(S_1 \cup S_2)$,

then $\exists \ v_1, \cdots, v_m \in S_1 \cup S_2$, such that

$v = a_1 v_1 + \cdots + a_m v_m$, where $a_1, \cdots, a_m \in \mathbb{F}$.

Without loss of generality, we can assume that

$v_1, \cdots, v_n \in S_1$, and $v_{n+1}, \cdots, v_m \in S_2$, where $0 \le n \le m$.

thus $\sum_{i=1}^{n} a_i v_i \in \text{span}(S_1)$ and $\sum_{i=n+1}^{m} a_i v_i \in \text{span}(S_2)$.

So $v = \sum_{i=1}^{n} a_i v_i + \sum_{i=n+1}^{m} a_i v_i \in \text{span}(S_1) + \text{span}(S_2)$.

Thus $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$.

"$\supseteq$": Method 1:

$\forall \ v \in \text{span}(S_1) + \text{span}(S_2)$, then $\exists \ v_1 \in \text{span}(S_1)$,

and $v_2 \in \text{span}(S_2)$, such that $v = v_1 + v_2$.

We write $v_1 = a_{11} v_{11} + \cdots + a_{1m} v_{1m}$, where $v_{1i} \in S_1$, $i = 1, \cdots, m$

$v_2 = a_{21} v_{21} + \cdots + a_{2n} v_{2n}$, where $v_{2i} \in S_2$, $i = 1, \cdots, n$.

Here $a_{1i}, a_{2j} \in \mathbb{F}$, $i = 1, \cdots, m$, $j = 1, \cdots, n$.

Then we have $v = v_1 + v_2 = \sum_{i=1}^{m} a_{1i} v_{1i} + \sum_{j=1}^{n} a_{2j} v_{2j}$

As $v_{11}, \cdots, v_{1m}, v_{21}, \cdots, v_{2n} \in S_1 \cup S_2$, we have that

$v \in \text{span}(S_1 \cup S_2)$. So $\text{span}(S_1 \cup S_2) \supseteq \text{span}(S_1) + \text{span}(S_2)$.

"$\supseteq$": Method 2.

For $i = 1, 2$, $S_i \subseteq S_1 \cup S_2 \Rightarrow \text{span}(S_i) \subseteq \text{span}(S_1 \cup S_2)$.

$$\Rightarrow \text{span}(S_1) + \text{span}(S_2) \subseteq \underbrace{\text{span}(S_1 \cup S_2) + \text{span}(S_1 \cup S_2)}_{= \text{span}(S_1 \cup S_2)}$$

Because $\text{span}(S_1 \cup S_2)$ is a subspace of V, which is closed under addition.

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15. Let S_1 and S_2 be subsets of a vector space V. Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$. Give an example in which $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal and one in which they are unequal.

Proof: Since $S_1 \cap S_2 \subseteq S_i$, $i = 1, 2$,

$$\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_i), \quad i = 1, 2$$

$$\Rightarrow \underbrace{\text{span}(S_1 \cap S_2) \cap \text{span}(S_1 \cap S_2)}_{= \text{span}(S_1 \cap S_2)} \subseteq \text{span}(S_1) \cap \text{span}(S_2)$$

"$=$": When $S_1 = S_2$, then $\text{span}(S_1 \cap S_2) = \text{span}(S_1)$ and $\text{span}(S_1) \cap \text{span}(S_2) = \text{span}(S_1)$

"$\neq$": Let $V = \mathbb{R}^2$ and $S_1 = \{(0,1)\}$, $S_2 = \{(0,2)\}$, then $S_1 \cap S_2 = \phi$, so $\text{span}(S_1 \cap S_2) = \{0\}$.

Note: $\text{span}(S)$ is defined to be the smallest subspace containing S, so if $S = \phi$, then $\text{span}(\phi) = \{0\}$.

While $\text{span}(\{(0,1)\}) = \{(0,y) : y \in \mathbb{R}\} = \text{span}(\{(0,2)\})$

so $\text{span}(S_1) \cap \text{span}(S_2) = \{(0,y) : y \in \mathbb{R}\}$
$\neq \text{span}(S_1 \cap S_2) = \{0\}$.

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7 Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbf{R}^2$.

Solution: Let $U = \{(n,n) : n \in \mathbb{Z}\}$, here $\mathbb{Z}$ denotes the set of all integers.

Then obviously U is closed under taking additive inverse, $-(n,n) = (-n,-n) \in U$

And $\forall\ (m,m), (n,n) \in U$,
$(m,m) + (n,n) = (m+n, m+n) \in U$.

But for $\lambda \in \mathbb{R}$ where λ is not any integer,
$\lambda(1,1) = (\lambda, \lambda) \notin U$.

So U is not a subspace of $\mathbb{R}^2$.

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8 Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

Solution: Let $U = \{(x,0), (0,y) : x, y \in \mathbb{R}\}$

then U is closed under scalar multiplication.

but for $(x,0), (0,y) \in U$ where $x \neq 0$, $y \neq 0$

the addition $(x,0) + (0,y) = (x,y) \notin U$.

So U is NOT a subspace of $\mathbb{R}^2$.

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21 Suppose

$$U = \{(x, y, x+y, x-y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

Solution: The zero vector $\vec{0}$ in $\mathbb{F}^5$ is $\vec{0} = (0,0,0,0,0)$.

For $x, y \in \mathbb{F}$, $(x, y, x+y, x-y, 2x)$

$$= (x, 0, x, x, 2x) + (0, y, y, -y, 0)$$
$$= x\underbrace{(1,0,1,1,2)}_{=v_1} + y\underbrace{(0,1,1,-1,0)}_{=v_2}$$

Thus v_1, v_2 are linearly independent and $U = \operatorname{span}\{v_1, v_2\}$.

So can we find $v_3, v_4, v_5 \in \mathbb{F}$, such that

$$\mathbb{F}^5 = \operatorname{span}(\{v_1, v_2\}) \oplus \operatorname{span}(\{v_3, v_4, v_5\}) \cdots (\star 1)$$

$(\star 1)$ means $\forall\ (b_1, \cdots, b_5) \in \mathbb{F}^5$, there exist unique $x_1, \cdots, x_5 \in \mathbb{F}$, such that

$$(b_1, \cdots, b_5) = \sum_{i=1}^{5} x_i v_i.$$

that is,

$(\star 2)$: $\begin{pmatrix} b_1 \\ \vdots \\ b_5 \end{pmatrix} = \begin{pmatrix} v_1^T & \cdots & v_5^T \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix}$ has unique solution.

So we can construct v_3, v_4, v_5 such that the 5×5 matrix $\begin{pmatrix} v_1 \\ \vdots \\ v_5 \end{pmatrix}$ is a triangular matrix, which can make $(\star 2)$ easy to solve. A simple one is to let

$$\begin{pmatrix} v_1 \\ v_2 \\ \boxed{\begin{matrix} v_3 \\ v_4 \\ v_5 \end{matrix}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 \\ \boxed{\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}} \end{pmatrix}$$

i.e. $v_3 = (0,0,1,0,0)$, $v_4 = (0,0,0,1,0)$, $v_5 = (0,0,0,0,1)$

And let $W = \operatorname{span}\{v_3, v_4, v_5\}$

$$= \{(0,0,x,y,z) : x,y,z \in \mathbb{F}\}.$$

then there holds $\mathbb{F}^5 = U \oplus W$.

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